

# Structural Complexity Of Quantum Networks

Michael Siomau<sup>1,2</sup>

<sup>1</sup>*Physics Department, Jazan University, P.O.Box 114, 45142 Jazan, Kingdom of Saudi Arabia*

<sup>2</sup>*Network Dynamics, Max Planck Institute for Dynamics and Self-Organization (MPIDS), 37077 Göttingen, Germany*

siomau@nld.ds.mpg.de

**Abstract.** Quantum network is a set of nodes connected with channels, through which the nodes communicate photons and classical information. Classical structural complexity of a quantum network may be defined through its physical structure, i.e. mutual position of nodes and channels connecting them. We show here that the classical structural complexity of a quantum network does not restrict the structural complexity of entanglement graphs, which may be created in the quantum network with local operations and classical communication. We show, in particular, that 1D quantum network can simulate both simple entanglement graphs such as lattices and random graphs and complex small-world graphs.

## INTRODUCTION

Modern world exhibits complex structures, which can be hardly tackled with simple linear models. Nonlinear networks [1], in contrast, can reliably describe a large portion of the structures and their dynamics. So far, networks have been successfully applied to describe such different natural and artificial processes as epidemic spreading [2], dynamics and self organization of neural circuits [3], intra-social interactions [4] and even autonomous control of robots [5] to name just a few. Networks permeate various structures, where different parties interact with each other. Differences in the characteristics of such interactions and how they evolve in time give growth to different types of structures: *regular* and *random* networks or, filling the gap between aforementioned, *complex* networks, which do not have a regular structure but neither are completely random [6].

While it is typically implied that networks obey the laws of classical physics, quantum networks have come into focus recently due to remarkable advances in quantum technologies. For instance, the entanglement-based quantum communication has been performed for distances over 144 km [7], which makes feasible the long distance quantum communication [8] and the quantum internet [9]. Also, latter achievements in experimental implementations of quantum computers [10] implies that quantum devices with limited computational capabilities could become available in the nearest future, which opens the possibility to realize the distributed quantum computation [11]. These emerging quantum systems are of non-trivial structure and need to be treated from the network perspective.

A study of a network, be it classical or quantum, always begins with defining its structure. Following classical paradigmatic viewpoint, structural complexity of quantum networks is typically associated with its physical structure, i.e. spatial position of nodes and their inter-connectivity due to channels. But, quantum mechanics as a non-local theory offers an intriguing possibility to prepare quantum systems located at physically disconnected nodes into an entangled state. This preparation can be accomplished with local operations and classical communication (LOCC) [12] without direct interaction of the distant nodes. This fact alone makes the difference between the physical structure of the quantum network (i.e. nodes and channels) and its entanglement structure (i.e. nodes and entanglement links). In this paper we aim to make the difference explicit.

To show the distinction between classical and quantum structural complexity of quantum networks, we shall consider the simplest possible 1D network configuration, where nodes are placed on a line at fixed distance from each other and are connected by channels. We assume that the nodes possess arbitrary number of qubits together with measurement devices and communicate with each other through the channels. For the sake of simplicity, we assume that only two-qubit entangled states can be created connecting two different neighboring nodes and no decoherence effects are taking place. We employ a particular type of LOCC – the entanglement swapping [13] – to show that non-

trivial entanglement graphs (*egraphs*) can be created on the 1D quantum network. In the next section, we introduce basic notations and the entanglement swapping operation. Then we show what kind of egraphs can be simulated on the 1D network and at what cost in terms of initial resources. We conclude in the last section putting forward open questions.

## ENTANGLEMENT SWAPPING

Exchanging photons and classical information, two qubits at neighboring nodes can be prepared in a two-qubit entangled state, which form an entanglement link between the nodes. Let us assume that the state is pure and thus may be written in the computational basis [12] as

$$|\varphi\rangle = \sqrt{\lambda_1}|00\rangle + \sqrt{\lambda_2}|11\rangle, \quad (1)$$

where  $\lambda_1$  and  $\lambda_2$  are the Schmidt coefficients conditioned by  $\lambda_1 \geq \lambda_2$  and  $\lambda_1 + \lambda_2 = 1$ .

If a node hosts two qubits from two different entangled pairs as depicted in Fig. 1a on the top, the qubits at the node may be measured in the Bell basis

$$|\Psi^\pm\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle), \quad |\Phi^\pm\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle), \quad (2)$$

projecting the initial four-qubit state  $|\psi_{1234}\rangle = |\varphi_{1,2}\rangle \otimes |\varphi_{3,4}\rangle$  onto one of the Bell states. This leads to "unification" of the two entanglement links into a single link disconnecting the host node as shown in Fig. 1a on the bottom.

In general, entanglement swapping reduces entanglement of the final state in comparison to the initial entanglement of the two entangled pairs [14]. But, Bose, Vedral and Knight showed that there is a measure of entanglement preserved *on average* under the entanglement swapping [15] for any pure state of qubits (1) – the singlet conversion probability defined as  $p = 2\lambda_2$ . This remarkable result allows us to assume that an arbitrary number of entangled links can be united into a single link by entanglement swapping with no detrimental effect to the entanglement of the final link. This assumption is crucial for further discussion and allows us to consider all the entanglement links in the egraph as equivalent, irrespectively whether they are created between neighboring (due to a photon exchange) or distant (due to entanglement swapping) nodes.

## QUANTUM EGRAPHS IN 1D QUANTUM NETWORK

It is evident from the Fig. 1a that the entanglement swapping operation "consume" two local entanglement links to create a single non-local entanglement link. Thus, construction of any non-trivial egraph on the 1D quantum network requires defined initial resources – a number of local entanglement links (which are created by sending a photon between two neighboring nodes). Let us assume that  $k$  entangled pairs are initially created between each pair of neighboring nodes. Let us find out how the total number of the initial entangled pairs in the network  $K = kN$  depends on the complexity of simulated egraphs.

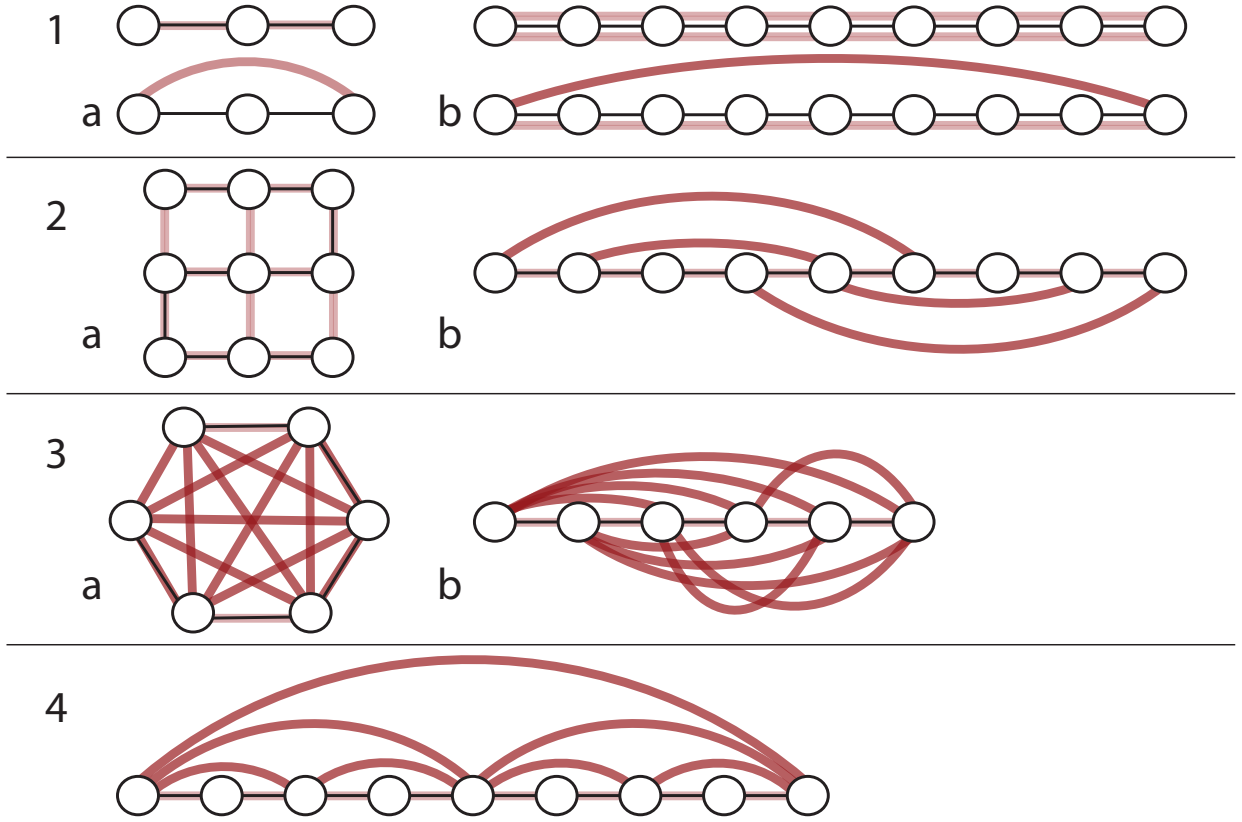
A ring can be simulated on the 1D network at cost of just  $K = 2N$  local entanglement links by applying entanglement swapping once at each node except of the borders (see Fig. 1b).

A 2D squared lattice of  $n \times n = N^2$  nodes (see Fig. 2) can be simulated on the 1D network at the cost of

$$K_{2D} = (n-1) \left[ \sum_{i=1}^{n-1} (2^i + 1) \right] + n^2 - 1 = (n-1) 2^n + 2(n-1)^2 = (\sqrt{N} - 1) 2^{\sqrt{N}} + 2(\sqrt{N} - 1)^2 \quad (3)$$

initial entangled pairs. The number of initial entangled pairs growth exponentially with the lattice size, which makes the simulation impractical.

This situation improves radically if we try to simulate a random graph with the 1D network. An arbitrary Erdos-Renyi random graph can be constructed from the complete graph [2], if the links are present/removed with some probability. If we can simulate the complete egraph on the 1D, i.e. the graph where every pair of nodes is connected by an entanglement link, then we can also simulate an arbitrary random egraph by destroying entanglement links, for



**FIGURE 1.** Structural complexity of 1D quantum network: black lines show the channels, red lines show entanglement links. 1a: Entanglement swapping at the central node connects the neighboring nodes with a non-local entanglement link. 1b: A ring is created from the 1D network by multiple entanglement swapping. 2: A 2D squared lattice of entanglement links: (2a) folded and (2b) planar representations of the egraph. 3: A complete egraph: (3a) folded and (3b) planar representations. 4: A small-world egraph.

example, by measuring one of the qubits from the entangled pair forming the entanglement link in the computational basis. The complete graph of  $N$  nodes (see Fig.3) can be created with

$$K_{RG} = \sum_{i=1}^{N-1} i(N-i) = \frac{(N-1)N(N+1)}{6} \quad (4)$$

initial entangled pairs. The number of initial entangled pairs growth polynomially with the number of nodes. This makes practical simulations feasible. Interestingly, because the complete graph of  $N$  nodes has exactly  $N(N-1)/2$  links, on average  $(N+1)/3$  local entanglement links are needed to construct an entanglement link of the complete egraph.

Finally, we would like to give an example of a small-world complex graph [16], which may be simulated on the 1D quantum network. Fig. 4 shows a hierarchical small-world egraph (which is also a fractal due to self-similarity) [17]. The construction of the egraph requires surprisingly small initial resources  $K_{H-SW} = N(1 + \log_2(N-1))$ , which makes it very useful in quantum communication due to its remarkable percolation properties [18].

## CONCLUSION

We showed a new feature of quantum networks – the inequivalence between classical and quantum structural complexity of the network. Already the 1D quantum network is able to simulate lattices and random and small-world graphs. The simulation of the 2D egraph requires exponential initial resources, thus is inefficient. In contrast, polynomial growth of the initial resources with respect to the number of nodes manifests that random and small-world egraphs can be simulated efficiently with 1D quantum network. This opens intriguing possibilities to study non-trivial egraphs and their dynamics with just a linear 1D setup.

Although our results are concerned with undirected egraphs, they may be extended to directed and weighted [6] egraphs. The direction may be introduced in the egraph model due to classical communication between the nodes by restricting some nodes from sending classical information to others. Such nodes become in-nodes, while those able to send classical information are out-nodes. The normalized weights of the entanglement links are essentially represented by the amount of entanglement preserved in the link. The singlet conversion probability of an entanglement link can be reduced by local operations or decoherence and increased by entanglement distillation [12].

General questions arise from our results: Can an arbitrary egraph be simulated on the 1D quantum network by LOCC? Under what conditions this simulation is efficient in terms of initial resources? Further studies are needed to answer the questions.

## ACKNOWLEDGMENTS

This work was supported by KACST under research grant no. 180-35.

## REFERENCES

- [1] S.N. Dorogovtsev and J.F.F. Mendes, *Evolution of Networks: From biological networks to the Internet and WWW* (Oxford University Press, 2003).
- [2] R. Pastor-Satorras, C. Castellano, P. Van Mieghem and A. Vespignani, Epidemic processes in complex networks, *Review of Modern Physics* **87**, 925 (2015).
- [3] S. Finger, *Origins of Neuroscience: A History of Explorations into Brain Function* (Oxford University Press, 2001).
- [4] D. Easley and J. Kleinberg, *Networks, Crowds, and Markets: Reasoning about a Highly Connected World* (Cambridge University Press, 2010).
- [5] S. Steingrube, M. Timme, F. Wörgötter and P. Manoonpong, Self-organized adaptation of a simple neural circuit enables complex robot behaviour, *Nature Physics* **6**, 224 (2010).
- [6] S. Boccaletti, V. Latorab, Y. Morenod, M. Chavezf and D.-U. Hwanga, Complex networks: Structure and dynamics, *Physics Reports* **424**, 175 (2006).
- [7] R. Ursin, F. Tiefenbacher, T. Schmitt-Manderbach, H. Weier, T. Scheidl, M. Lindenthal, B. Blauensteiner, T. Jennewein, J. Perdigues, P. Trojek, B. Ömer, M. Fürst, M. Meyenburg, J. Rarity, Z. Sodnik, C. Barbieri, H. Weinfurter and A. Zeilinger, Entanglement-based quantum communication over 144 km, *Nature Physics* **3**, 481 (2007).
- [8] L.-M. Duan, M.D. Lukin, J.I. Cirac and P. Zoller, Long-distance quantum communication with atomic ensembles and linear optics, *Nature* **414**, 413 (2001).
- [9] H.J. Kimble, The quantum internet, *Nature* **453**, 1023 (2008).
- [10] T.D. Ladd, F. Jelezko, R. Laflamme, Y. Nakamura, C. Monroe and J.L. O'Brien, Quantum computers, *Nature* **464**, 45 (2010).
- [11] J.I. Cirac, A.K. Ekert, S.F. Huelga and C. Macchiavello, Distributed quantum computation over noisy channels, *Phys. Rev. A* **59**, 4249 (1999).
- [12] M.A. Nielsen and I.J. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, 2000).
- [13] J.-W. Pan, D. Bouwmeester, H. Weinfurter and A. Zeilinger, Experimental Entanglement Swapping: Entangling Photons That Never Interacted, *Physical Review Letters* **80**, 3891 (1998).
- [14] S. Perseguers, J.I. Cirac, A. Acin, M. Lewenstein and J. Wehr, Entanglement distribution in pure-state quantum networks, *Physical Review A* **77**, 022308 (2008).

- [15] S. Bose, V. Vedral and P.I. Knight, Purification via entanglement swapping and conserved entanglement, *Physical Review A* **60**, 194 (1999).
- [16] D.J. Watts and S.H. Strogatz, Collective dynamics of 'small-world' networks, *Nature* **393**, 440 (1998).
- [17] S. Boettcher, J.L. Cook and R.M. Ziff, Patchy percolation on a hierarchical network with small-world bonds, *Physical Review E* **80**, 041115 (2009).
- [18] M. Siomau, Quantum Entanglement Percolation.